Paper ???

©Civil-Comp Press, 2012
Proceedings of the Eleventh International Conference
on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Axial Shear Force and Bending Moment Interaction in Elastoplastic Analysis of Steel Frames

M.M.S. Manola and V.K. Koumousis
Institute of Structural Analysis & Aseismic Research
National Technical University of Athens, Greece

Abstract

In this paper limit elastoplastic analysis of frame structures with hardening behaviour and axial-shear force–bending moment interaction is examined in the framework of mathematical programming. The maximum load carrying capacity of the structure is determined by solving an optimization problem with linear equilibrium, compatibility and yield constraints together with a complementarity constraint that is treated using the penalty function method. Incorporation of the shear force effect at yield determines a nonlinear three-dimensional yield surface that is linearized with appropriate polyhedra. The proposed method is implemented for the analysis of steel frames for rigid-perfectly plastic and isotropic hardening behaviour. Numerical results are compared to those of axial force-bending moment interaction underlining the significant role of shear force effect especially for shorter members.

Keywords: limit analysis, holonomic constraints, complementarity, stress resultant interaction, mathematical programming.

1 Introduction

The determination of the ultimate limit state of a structure at incipient collapse subject to constant and monotonically varying loading i.e. finding the maximum load that a structure can sustain is the goal of elastoplastic analysis. In this context, limit analysis represents a computationally efficient method since the ultimate load is assessed by considering only the final stage of plastic collapse without analyzing the entire history of response following a step by step method. In the past decades, limit analysis has been successfully imbedded into the framework of mathematical programming offering a unifying approach for discrete problems of structural plasticity.

The ultimate load-carrying capacity of the structure is evaluated by solving an optimization problem based on piecewise linear constitutive relations following associated flow rules. The formulation of this optimization problem, pioneered by Maier et al. [1-5], constitutes a combination of limit and deformation analysis under holonomic or nonholonomic assumption. This means that the maximum load factor is assessed at the final stage of collapse under equilibrium, yield and deformation i.e. compatibility constraints. The nonholonomic assumption accounts for path-dependence (irreversibility) of plastic deformations due to elastic unloadings-reloadings approximating the realistic structural behaviour. On the other hand, holonomy is based on the assumption of negligible effects of local unloading and represents a path-independent (reversible) structural behaviour. The latter assumption is considered practically valid in describing the plastic behaviour of structures under the presence of proportionally and monotonically increasing external actions [4].

Linear programming has been extensively used for limit analysis of frames by assuming piecewise linear yield surfaces and rigid-perfectly plastic behaviour. Quadratic programming as an extension of linear programming has been utilized to account for rigid-perfectly plastic and hardening structural behaviour [1, 2]. The exploration of the complementarity problem by Cottle [6] has directed the treatment of elastoplastic analysis in the form of a Parametric Linear Complementarity Problem (De Donato and Maier [4]), while Kaneko later proposed an efficient reformulation of this problem [7]. Moreover, a variety of alternative mathematical programming procedures such as iterative Linear Programming, Quadratic Programming, Restricted Basis Linear Programming, Parametric Linear Complementarity and Parametric Quadratic Programming procedures have been applied and compared by Maier, Grierson and Best [8] for elastoplastic analysis of structures.

The recent development of mathematical programming algorithms appropriate for Mathematical Programming with Equilibrium Constraints (MPEC) problems [9] has extended the potential of the proposed methods for structural analysis for both holonomic and nonholonomic assumptions [10-14]. In this context, softening structural behaviour has been also thoroughly examined under the effect of combined stresses (axial force-bending moment) by Cocchetti and Maier [15], Tangaramvong and Tin-Loi [16, 17].

The aim of this work is to examine the shear force effect on the ultimate load carrying capacity of frame structures. For this reason, an optimization problem with equilibrium, compatibility, yield and complementarity constraints on the basis of holonomic assumption is formulated. The adopted yield criterion accounts for the axial-shear force-bending moment interaction and the proposed formulation is applied for rigid-perfectly plastic and isotropic hardening behaviour.

The organization of the paper is as follows. First, the governing relations of holonomic elastoplastic problem based on equilibrium, kinematical and constitutive relations are summarized. Then, the formulation of the limit analysis as a MPEC problem is presented incorporating the axial-shear force-bending moment interaction. Subsequently numerical examples of steel frames are presented that illustrate the applicability of the proposed method and the role of shear force effect on the maximum load factor.

2 Governing relations

Limit analysis with mathematical programming methods aims at computing the safety factor of a structure based on several assumptions [1,15,17]. First, plane frames are considered to consist of straight prismatic elements subjected herein only to nodal loading for reasons of simplicity. Moreover, frame displacements are assumed small enough so that the equilibrium equations refer to the initial undeformed configuration. In addition plastic hinges are considered formed only at critical sections, whereas the remaining parts behave elastically. The local nonlinear behaviour of critical sections is described by a piecewise linear model and yield functions are linearized appropriately. Furthermore, under monotonically increasing external loading, local unloading if happens, is assumed reversible. Thus a holonomic i.e. path-independent structural behaviour is adopted.

2.1 Equilibrium

From the six independent actions at the ends of a plane beam element, by applying the three equilibrium equations three may be considered as independent and the remaining as dependent end actions. Herein, the axial force (s_1^i), bending moment at the start node *j* (s_2^i) and bending moment at the end node *k* (s_3^i), as shown in Figure 1, in red color, are considered as independent. The structural equilibrium relationship for the whole structure is then established as:

$$B \cdot s = F \tag{1}$$

where B is the ($nf \times 3nel$) structural equilibrium matrix, formed by assembling the corresponding element equilibrium matrices, s is a ($3nel \times 1$) vector for all primary stress resultants and F is a ($nf \times 1$) matrix of nodal external loading. It is noted that *nel* denotes the number of elements, *nf* the number of degrees of freedom. The latter can be expressed as:

$$F = a \cdot f + f_d \tag{2}$$

where a is a scalar load factor, f is the basic monotonically varying nodal forces and f_d is the fixed nodal vector.

Vertical dead and live loads are usually considered as constant, while lateral loads that account for earthquake or wind loading are proportionally increased until an ultimate state or incipient collapse.

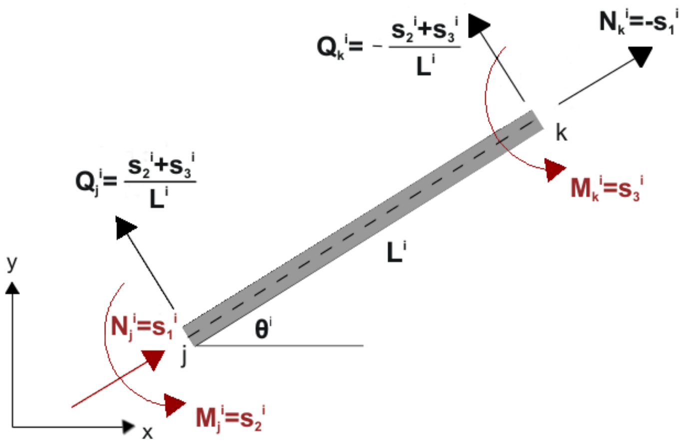

Figure 1: Frame element i with positive stress resultants

2.2 Compatibility

Compatibility conditions relate the member deformation q^i to the nodal displacements u^i. Since small displacements are considered in this work, the compatibility condition for the whole structure is given by the following linear congruent relation:

$$q = B^T \cdot u \tag{3}$$

where q is the $(3nel \times 1)$ strain vector and u is the $(nf \times 1)$ nodal displacement vector.

2.3 Strain decomposition

The constitutive law that governs the behaviour of a generic element is based on strain decomposition into the elastic and plastic component, as depicted in Figure 2. For the entire structure this is expressed as:

$$q = e + p \tag{4}$$

where q is the total strain, e is the elastic and p the plastic strain.
The elastic branch is fully described by the relation:

$$s = S \cdot e \tag{5}$$

where S is the $(3nel \times 3nel)$ stiffness matrix of the structure.
Following the notions of associative plasticity, plastic deformations p are defined in view of the holonomic assumption as follows:

$$p = N \cdot z \tag{6}$$

where N is the diagonal ($3nel \times ynel$) matrix of all unit normals to the yield hyperplanes, z is the ($2ynel \times 1$) of plastic multipliers and y the number of yield hyperplanes at each element end.

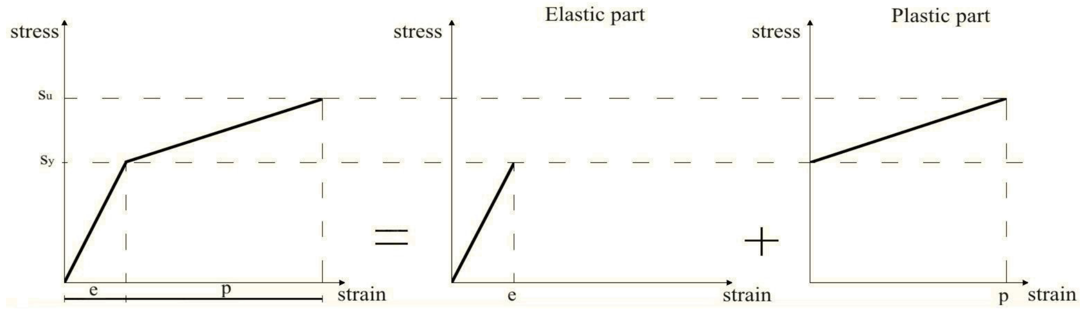

Figure 2: Strain decomposition into elastic and plastic components

2.4 Yield condition

As far as the yield conditions are concerned, plastic hinges are considered to develop at the element's ends under the axial-shear-bending interaction. Various yield criteria exist in the literature [18] for different materials and/or cross sectional shapes; herein the generalized Gendy-Saleeb yield criterion given by the following relation is adopted [19]:

$$\Phi = n^2 + v^2 + \frac{1}{\lambda} \cdot m^2 - 1 \tag{7}$$

where $n = \frac{s_1^i}{s_{1y}^i}$, $v = \frac{s_2^i + s_3^i}{L^i \cdot v_y^i}$, $m = \frac{s_2^i}{s_{2y}^i}$ or $m = \frac{s_3^i}{s_{3y}^i}$ and s^i_{1y}, v^i_y, s^i_{2y} and s^i_{3y} are the individual axial, shear and moment plastic capacities for the critical cross sections of the elements. It is noted that the above yield relation is valid for both rectangular and wide flange-I cross-sections. The introduced shape dependent parameter λ is evaluated for rectangular cross-sections and I-sections respectively using the following relations:

$$\lambda = 1 - n^2, \quad \lambda = 1 - 1.1|n| \tag{8}$$

In this work, the aforementioned yield criterion is represented by a 3d nonlinear surface that is approximated by using 32 plane triangles (16 for $m>0$ and 16 for $m<0$) as shown in Figure 3. This converts the yield condition into a set of linear ones which are advantageous for the mathematical programming formulation of the problem.

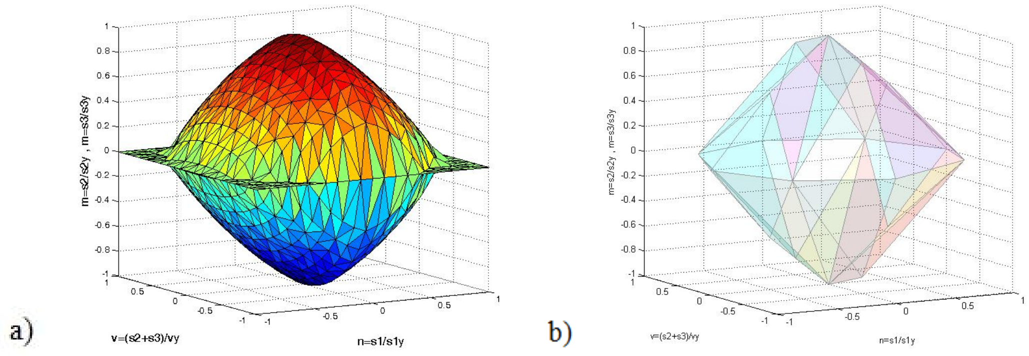

Figure 3: a) Nonlinear Gendy-Saleeb yield criterion and b) Linearized yield criterion

More specifically, the approximation of the 32 plane triangles corresponds to 32 equations for the corresponding planes in x, y, z space, which are of the following form:

$$Ax + By + Cz + D = 0 \tag{8}$$

where A,B,C are the components of the normal vector of the plane and –D the distance of the plane from the origin.

Using n_i, v_i, m_i instead of x, y, z and performing an appropriate manipulation, the following forms of these equations are systematically determined for start nodes *j* (Equation 10) and end nodes *k* (Equation 11) with respect to $s_1^i,\ \ s_2^i,\ \ s_3^i$:

$$\begin{gathered} A_t \cdot n^i + B_t \cdot v^i + C_t \cdot m^i + D_t = 0 \Leftrightarrow A_t \cdot \frac{s_1^i}{s_{1y}^i} + B_t \cdot \frac{s_2^i + s_3^i}{L^i \cdot v_y^i} + C_t \cdot \frac{s_2^i}{s_{2y}^i} + D_t = 0 \Leftrightarrow \\ \frac{A_t}{C_t} \cdot np_{21}^i \cdot s_1^i + \frac{B_t}{L^i \cdot C_t} \cdot np_{2v}^i \cdot s_2^i + \frac{B_t}{L^i \cdot C_t} \cdot np_{2v}^i \cdot s_3^i + s_2^i = -\frac{D_t}{C_t} \cdot s_{2y}^i \Leftrightarrow \\ Ac_t \cdot np_{21}^i \cdot s_1^i + \left(Bc_t \cdot np_{2v}^i + 1 \right) \cdot s_2^i + Bc_t \cdot np_{2v}^i \cdot s_3^i = -Dc_t \cdot s_{2y}^i \end{gathered} \tag{9}$$

$$\begin{gathered} A_t \cdot n^i + B_t \cdot v^i + C_t \cdot m^i + D_t = 0 \Leftrightarrow A_t \cdot \frac{s_1^i}{s_{1y}^i} + B_t \cdot \frac{s_2^i + s_3^i}{L^i \cdot v_y^i} + C_t \cdot \frac{s_3^i}{s_{3y}^i} + D_t = 0 \Leftrightarrow \\ \frac{A_t}{C_t} \cdot np_{31}^i \cdot s_1^i + \frac{B_t}{L^i \cdot C_t} \cdot np_{3v}^i \cdot s_2^i + \frac{B_t}{L^i \cdot C_t} \cdot np_{3v}^i \cdot s_3^i + s_3^i = -\frac{D_t}{C_t} \cdot s_{3y}^i \Leftrightarrow \\ Ac_t \cdot np_{31}^i \cdot s_1^i + Bc_t \cdot np_{3v}^i \cdot s_2^i + \left(Bc_t \cdot np_{3v}^i + 1 \right) \cdot s_3^i = -Dc_t \cdot s_{3y}^i \end{gathered} \tag{10}$$

where A_t, B_t, C_t and D_t the coefficients of each plane equation, $Ac_t = \frac{A_t}{C_t},\ Bc_t = \frac{B_t}{L^i \cdot C_t},\ Dc_t = \frac{D_t}{C_t},\quad np_{21}^i = \frac{s_{2y}^i}{s_{1y}^i},\ np_{31}^i = \frac{s_{3y}^i}{s_{1y}^i},\ np_{2v}^i = \frac{s_{2y}^i}{v_y^i},\ np_{3v}^i = \frac{s_{3y}^i}{v_y^i}$, *i=1,…,nel* (number of elements) and *t=1,…,16* (m>0). It is noted that for *t=17,..,32* (m<0) the plane equations for each element end are given by:

$$-Ac_t \cdot np_{21}^i \cdot s_1^i - \left(Bc_t \cdot np_{2v}^i + 1\right) \cdot s_2^i - Bc_t \cdot np_{2v}^i \cdot s_3^i = Dc_t \cdot s_{2y}^i \qquad (11)$$

$$-Ac_t \cdot np_{31}^i \cdot s_1^i - Bc_t \cdot np_{3v}^i \cdot s_2^i - \left(Bc_t \cdot np_{3v}^i + 1\right) \cdot s_3^i = Dc_t \cdot s_{3y}^i \qquad (12)$$

The first part of both equations (10) and (11) represent the total stress state of a yielded element end under the effect of axial-shear force-bending moment interaction. Thus, the coefficients that multiply the variables s_1^i, s_2^i, s_3^i form the ($3\times 2ynel$) matrix N^i for each element, while the second part of the equations forms the $(2ynel\times 1)$ vector r^i of the yield limits as follows:

$$N^{iT} = \begin{bmatrix} Ac_1 \cdot np_{21}^i & Bc_1 \cdot np_{2v}^i + 1 & Bc_1 \cdot np_{2v}^i \\ \vdots & \vdots & \vdots \\ Ac_{16} \cdot np_{21}^i & Bc_{16} \cdot np_{2v}^i + 1 & Bc_{16} \cdot np_{2v}^i \\ -Ac_{17} \cdot np_{21}^i & -Bc_{17} \cdot np_{2v}^i - 1 & -Bc_{17} \cdot np_{2v}^i \\ \vdots & \vdots & \vdots \\ -Ac_{32} \cdot np_{21}^i & -Bc_{32} \cdot np_{2v}^i - 1 & -Bc_{32} \cdot np_{2v}^i \\ Ac_1 \cdot np_{31}^i & Bc_1 \cdot np_{3v}^i & Bc_1 \cdot np_{3v}^i + 1 \\ \vdots & \vdots & \vdots \\ Ac_{16} \cdot np_{31}^i & Bc_{16} \cdot np_{3v}^i & Bc_{16} \cdot np_{3v}^i + 1 \\ -Ac_{17} \cdot np_{31}^i & -Bc_{17} \cdot np_{3v}^i & -Bc_{17} \cdot np_{3v}^i - 1 \\ \vdots & \vdots & \vdots \\ -Ac_{32} \cdot np_{31}^i & -Bc_{32} \cdot np_{3v}^i & -Bc_{32} \cdot np_{3v}^i - 1 \end{bmatrix}, \quad r^i = \begin{bmatrix} -Dc_1 \cdot s_{2y}^i \\ \vdots \\ -Dc_{16} \cdot s_{2y}^i \\ Dc_{17} \cdot s_{2y}^i \\ \vdots \\ Dc_{32} \cdot s_{2y}^i \\ -Dc_1 \cdot s_{3y}^i \\ \vdots \\ -Dc_{16} \cdot s_{3y}^i \\ Dc_{17} \cdot s_{3y}^i \\ \vdots \\ Dc_{32} \cdot s_{3y}^i \end{bmatrix} \qquad (13)$$

The linearized yield surface is assumed to follow an isotropic hardening rule [16,17,20], i.e. the yield surface expands about the origin retaining its similarity. This means that an already yielded point of a given stress state ($N^{iT} \cdot s^i = r^i$, Figure 4a) moves along the direction of the scaled plastic strain $a_t{\cdot}z_t$, where a_t is the scaling factor and z_t is the plastic multiplier, as shown in Figure 4b. It is noted that p_c is the assumed critical plastic strain, $h^i = \dfrac{s_{2u}^i - s_{2y}^i}{p_c}$, $s_{2u}^i = q_0^i \cdot s_{2y}^i$, where q_0^i is the corresponding overstrength factor of the element. Thus the ($2y\times 2y$) hardening matrix H^i for each element is determined as:

$$H^i = h^i \cdot \begin{bmatrix} -Dc_1 \cdot a_1 & -Dc_1 \cdot a_2 & \dots & -Dc_1 \cdot a_{32} & 0 & 0 & \dots & 0 \\ -Dc_2 \cdot a_1 & -Dc_2 \cdot a_2 & \dots & -Dc_2 \cdot a_{32} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 & \dots & 0 \\ Dc_{32} \cdot a_1 & Dc_{32} \cdot a_2 & \dots & Dc_{32} \cdot a_{32} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -Dc_1 \cdot a_1 & -Dc_1 \cdot a_2 & \dots & -Dc_1 \cdot a_{32} \\ 0 & 0 & \dots & 0 & -Dc_2 \cdot a_1 & -Dc_2 \cdot a_2 & \dots & -Dc_2 \cdot a_{32} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & Dc_{32} \cdot a_1 & Dc_{32} \cdot a_2 & \dots & Dc_{32} \cdot a_{32} \end{bmatrix} \qquad (14)$$

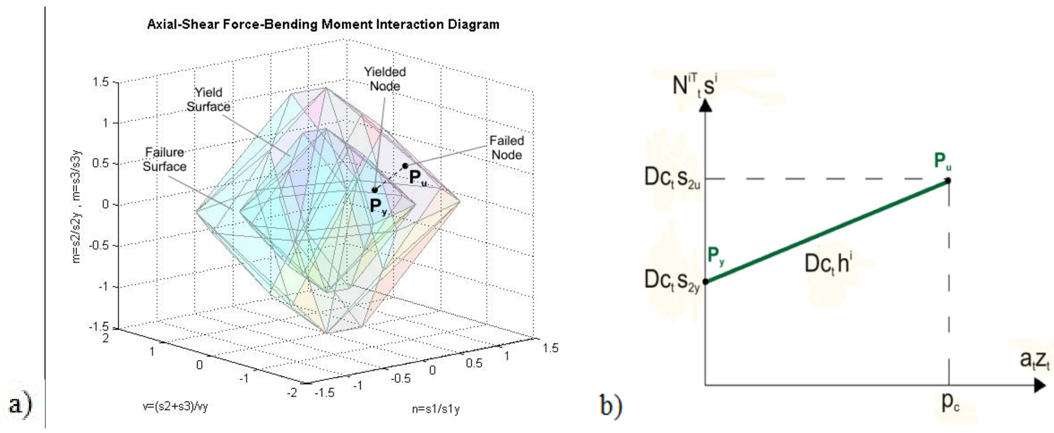

Figure 4: a) Stress condition of a yielded node and b) corresponding isotropic hardening behaviour

Finally, the diagonal matrices *N (3nel×2ynel)* and *H (2ynel×2ynel)* and the vector *r (2ynel×1)* for the whole structure are assembled and the yield conditions for the whole structure are formed as a *(2ynel×1)* non negative vector *w* as follows*:*

$$w = -N^T \cdot s + H \cdot z + r \geq 0 \tag{15}$$

In Figure 5, the geometrical interpretation of yielding is depicted. OR_1 and OR_2 are the distances from the origin O (0, 0, 0) to each plane triangle and denote the yield limits that correspond to each yield hyperplane. Stress points P_1 and P_2 are projected on OR_1 and OR_2 as P_1' and P_2' respectively. The yield conditions are then expressed as: $OP_1' \leq OR_1$ and $OP_2' \leq OR_2$. In case of rigid-perfectly plastic behaviour the aforementioned relations can be expressed as $N^{iT} \cdot s^i \leq r^i$, while in the case of isotropic hardening $N^{iT} \cdot s^i \leq r^{i\prime}$, where $r^{i\prime} = r^i + H^i \cdot z^i$.

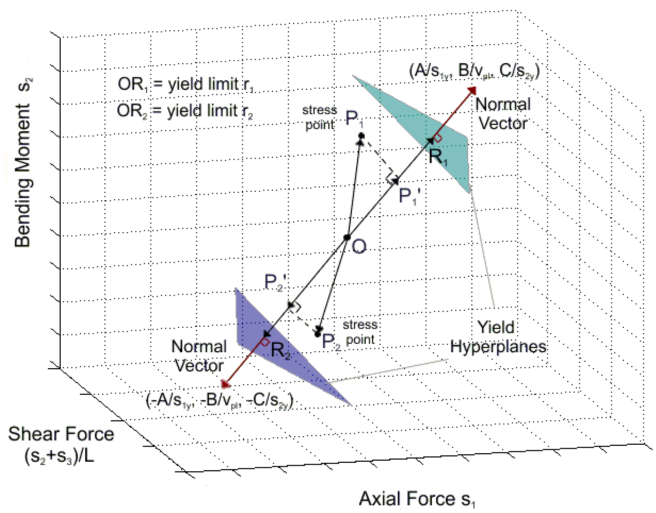

Figure 5: Geometrical interpretation of yielding

2.5 Complementarity condition

An additional constraint that regulates elastoplastic behaviour of the structure is the complementarity condition:

$$w^T \cdot z = 0, \quad w \geq 0, \quad z \geq 0 \quad or \quad w_i z_i = 0 \quad i = 1, 2, ..., n \tag{16}$$

The aforementioned condition prohibits simultaneous activation of plastic deformation and unloading. More specifically, the complementarity condition indicates that when the yield function w_t is activated ($w_t=0$), the corresponding plastic multiplier z_t should be greater than zero. Similarly, when the yield hyperplane t is inactive ($w_t>0$), the corresponding plastic multiplier $z_t=0$, indicating that no plastic flow occurs.

3 Mathematical programming and limit analysis

Equations (1)-(6), (16) and (17) formulate a holonomic elastoplastic problem that describes the whole structural path-independent behaviour. The evolution of plastic behaviour throughout the whole structure under the action of several loads follows the limitations imposed by static, kinematic, constitutive law and the additional complementarity condition. More specifically, a statically admissible state is ensured by equilibrium and yield conditions (Equations 1 and 16), the kinematically admissible state is enforced by compatibility relation (Equation 3), whereas the role of constitutive law and complementarity requirements are introduced by Equations (4)-(6) and Equation (17) respectively. Therefore, the elastoplastic holonomic problem can be expressed as:

$$\left.\begin{array}{l} B \cdot s = a \cdot f + f_d \\ q = B^T \cdot u \\ q = e + p \\ s = S \cdot e \\ p = N \cdot z \\ w = -N^T \cdot s + H \cdot z + r \\ w^T \cdot z = 0, \quad w \geq 0, \quad z \geq 0 \end{array}\right\} \tag{17}$$

The above system of equations can be simplified by retaining as unknown the variables s, u, z so that a Mixed Complementarity Problem (MCP) is formulated. This is equivalently converted into the following optimization problem, the solution of which provides simultaneously the load multiplier a, the corresponding stresses s and displacements u together with the plastic multipliers z [16,17]:

$$\left.\begin{array}{ll} \text{maximize} & a \\ \text{subject to} & B \cdot s - a \cdot f = f_d \\ & S^{-1} \cdot s - B^T \cdot u + N \cdot z = 0 \\ & w = -N^T \cdot s + H \cdot z + r \geq 0, \quad z \geq 0 \\ & w^T \cdot z = 0 \end{array}\right\} \tag{18}$$

Mathematically this is a nonconvex optimization problem that is known as a Mathematical Programming with Equilibrium Constraints (MPEC) problem including the complementarity constraint that acts as a multi-switch and is of discrete rather than continuous nature. This disjunctive constraint is difficult to handle numerically leading to numerical instabilities due to lack of convexity and smoothness. Despite all these inherent difficulties, the MPEC problem (19) can be solved by converting it into a standard, though still nonconvex, nonlinear programming problem (NLP) by suitably treating the complementarity condition. Several techniques have been proposed such as penalty function formulation, relaxation method, active set identification approach, sequential quadratic programming (SQP), and interior point methods among others [21]. Herein, the penalty function approach is adopted. According to this the complementarity constraint is handled in the objective function by a parametric reformulation, in which an increased value of the parameter exerts a pressure on the complementarity condition leading it to vanish. This formulation is as follows:

$$\left.\begin{array}{ll} \text{maximize} & a - \rho \cdot w^T \cdot z \\ \text{subject to} & B \cdot s - a \cdot f = f_d \\ & S^{-1} \cdot s - B^T \cdot u + N \cdot z = 0 \\ & w = -N^T \cdot s + H \cdot z + r \geq 0, \quad z \geq 0 \end{array}\right\} \tag{19}$$

It is worth noting that the above formulation is sensitive to the initial values of ρ and its subsequent increase. Typical starting values of ρ are between 0.1 and 1 with an update rule of $\rho=10\rho$ after each NLP solution until an appropriate convergence tolerance is reached ($w^T z \leq 10^{-6}$) [16,17].

For the case of the rigid-perfectly plastic behaviour the formulation of the above problem is reduced to the Linear Programming - LP problem:

$$\left.\begin{array}{ll} \text{maximize} & a \\ \text{subject to} & B \cdot s - a \cdot f = f_d \\ & N^T \cdot s \leq r \end{array}\right\} \tag{20}$$

The decision variables of this optimization problem are the load factor a and the element stresses s. The aforementioned formulation (21) constitutes the static approach of limit analysis and offers a lower bound of the load factor a for plastic collapse of the structure. On the grounds of LP, if the solution of the static LP problem coincides with the solution of the kinematic LP problem it provides the true collapse loading factor *a*. This uniqueness of the collapse load factor stems from the strong duality theorem of LP [8, 22]. Moreover, it is noted that most LP solvers, including the one of Matlab optimization toolbox, provide also the Lagrange multipliers of the optimal solution that combine directly the variables of static problem (primal) with the ones of kinematic (dual). Thus, by solving only the optimization problem (21), apart from variables *a* and *s*, the variables *u* (displacements) and *z* (plastic multiplier rates) can be obtained.

4 Examples

The limit analysis problem described in relations (20) and (21) is implemented in Matlab code for the analysis of steel frame structures. The data are processed by *fmincon* solver (appropriate for the minimization of constrained nonlinear multivariable function), with the interior-point algorithm selected as optimization method. The aim is to investigate the role of combined axial-shear force-bending moment interaction and its influence on structural behaviour. For this purpose, the steel frames of Figure 6, having the material and mechanical properties presented in Table 1, have been examined for axial force-bending moment interaction (NM interaction) and for axial-shear force-bending moment interaction (NQM interaction) for the cases of rigid-perfectly plastic and elastoplastic isotropic hardening behaviour. It is noted that results for axial force-bending moment interaction are determined by solving optimization problem (20) with a hexagonal yield criterion appropriate for I-steel cross-sections [16]. Results for all analysis cases are presented in Table 2, while maximum load differences for each variant are shown in Table 3.

As expected the load carrying capacity of the structure is reduced for NQM interaction as compared to NM interaction due to the effect of shear force. Moreover, it is noted that under the same conditions i.e. NM or NQM interaction, isotropic hardening consideration provides greater values for maximum load *α* as compared to rigid-perfectly plastic behaviour for both frames, as anticipated.

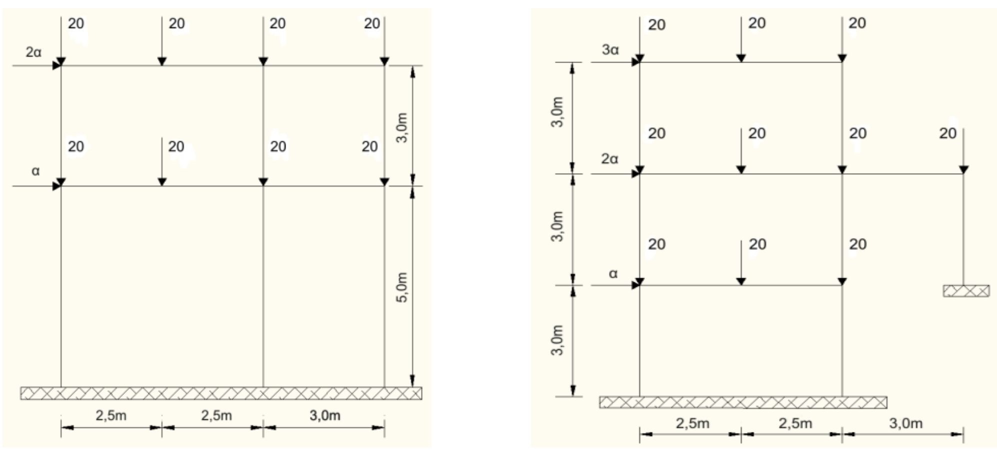

Figure 6: Steel frames that have been analyzed

Material Properties & Geometrical Characteristics	FRAME 1	FRAME 2
	Columns (HE400A)	Beams (IPE200)
Yield limit of axial force s_{1y}	3736.5 kN	669.3 kN
Ultimate limit of axial force s_{1u}	5604.8 kN	803.2 kN
Yield limit of shear force v_y	778.1 kN	189.9 kN
Ultimate limit of shear force v_u	1167.2 kN	284.9 kN
Yield limit of bending moment s_{2y}, s_{3y}	602.1 kNm	51.8 kNm
Ultimate limit of bending moment s_{2u}, s_{3u}	903.1 kNm	62.2 kNm
Hardening *h*	10034.5 kNm	432 kNm
Number of elements *nel*	12	14
Number of nodes *nnodes*	11	13
Number of degrees of freedom *nf*	24	30
Modulus of elasticity *E*	$2 \cdot 10^8$ kN/m²	

Table 1: Properties of frames under examination

	NM Interaction (rigid-p.plastic)	NM Interaction (hardening)	NQM Interaction (rigid-p.plastic)	NQM Interaction (hardening)
Frame 1	155.35	165.49	131.55	142.97
Frame 2	67.57	79.83	59.73	71.60

Table 2: Maximum Load *a* for all analysis cases

	$(a_{NQM} - a_{NM})/a_{NQM}$ %	
	rigid-p.plastic	hardening
Frame 1	-18.09%	-15.75%
Frame 2	-13.12%	-11.49%

Table 3: Difference of maximum load *a* for NM and NQM interaction.

In Figure 7, NM and NQM interaction diagrams for frame 1 are presented. Black and blue spots correspond to start and end element nodes respectively. It is evident that for both types of interaction the role of bending moment is dominant, while beam cross-sections yield mainly under the combined action of axial force and bending moment due to the presence of the external lateral load. However, for the case of NQM interaction the role of shear force effect is more evident at shorter columns (Figure 7b).

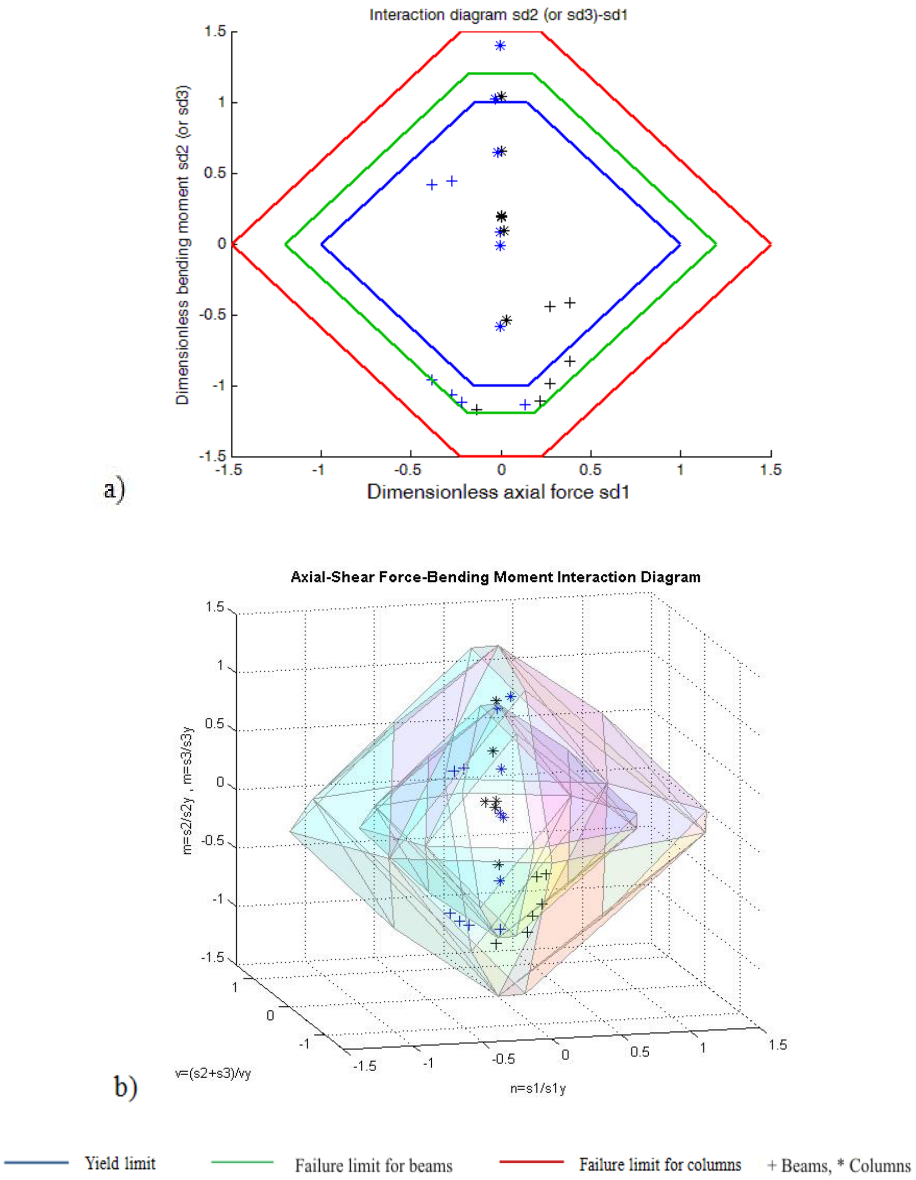

Figure 7: (a) NM interaction diagram and (b) NQM interaction diagram for frame 1

The vertices of the state of all sections of the frame are arranged within the yield and failure surfaces depending on the connectivity, strength and loading. Some of them remain elastic, others have yielded and some others have reached the failure surface. In the attempt to trace the evolution of the critical sections, the convex hull surface, that embraces all vertices determining the lines of defense of the structure as monotonic loading progresses, is determined and is proved quite informative. In Figure 8, the evolution of the convex hull of the dominant stress states of the cross-sections of frame 1 for increasing loading levels is presented. It is evident that under monotonically increasing loading the stress points that form the convex hull, i.e. the outer convex surface of all vertices, at the first loading stage, dominate and lead the plastic behaviour usually until collapse.

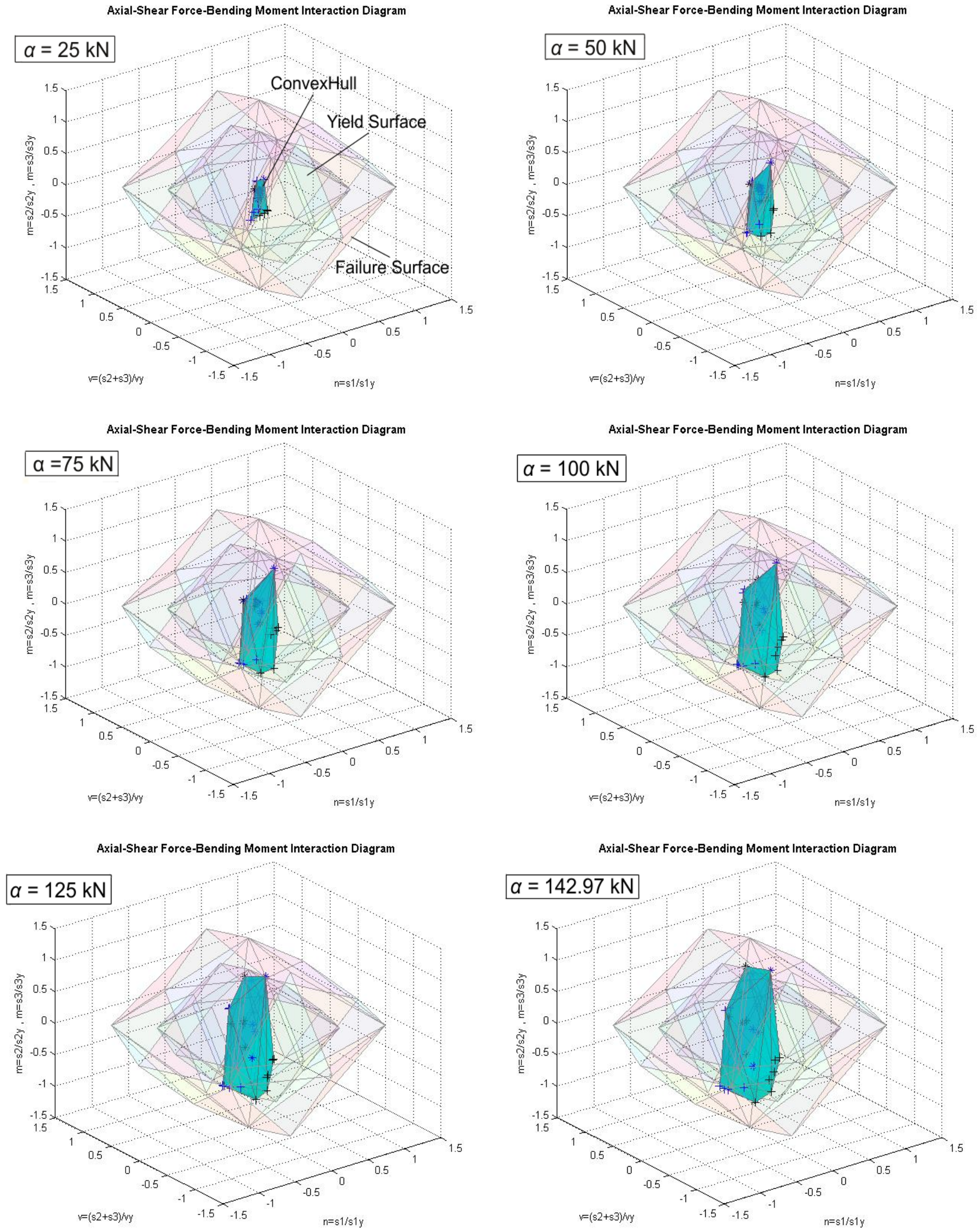

Figure 8: Evolution of convex hull for increasing applied load for frame 1

The deformed shapes of frame 2 under NQM interaction consideration for isotropic hardening and rigid-perfectly plastic behaviour are illustrated in Figure 9. It is noted that red and green plastic hinges denote negative and positive yielding respectively. Due to unbounded ductility, greater displacements are determined and more plastic hinges are formed for rigid-perfectly plastic assumption. Figures 10 and 11 depict the convex hulls of the whole structure and separately those corresponding to beams and columns of frame 2. It is evident that due to the action of lateral loading and the absence of a diaphragm, beams are mainly subject to the axial-bending moment interaction, while the shear force interaction is more intense at column cross-sections.

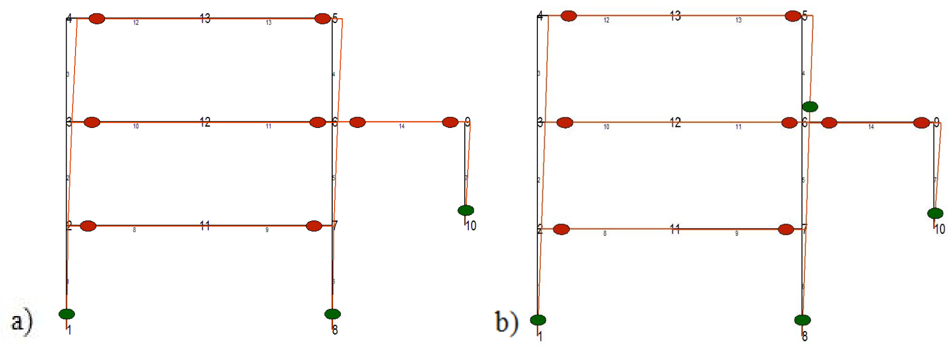

Figure 9: Deformed shape under NQM interaction consideration for (a) hardening behaviour and (b) rigid-p.plastic behaviour.

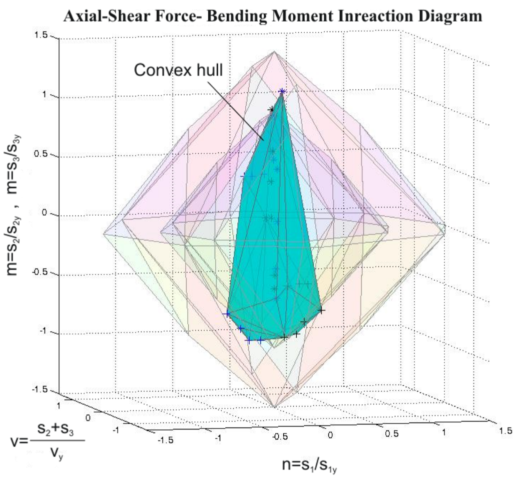

Figure 10: Convex hull of frame 2

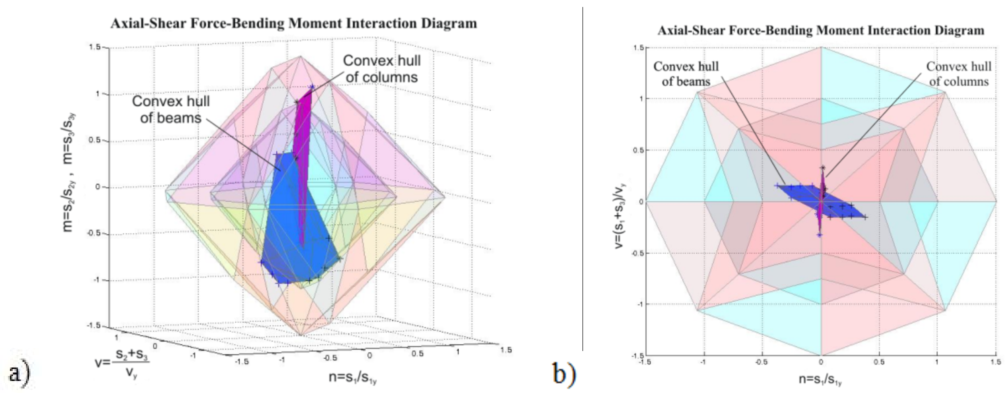

Figure 11: Convex hulls of beams and columns of frame 2

5 Concluding remarks

The limit analysis under holonomic behaviour and mathematical programming offers a broader perspective for an elastoplastic analysis of structures incorporating hardening behaviour and stress resultant interaction. The maximum load carrying capacity of a structure is established by solving an optimization problem subjected to constraints that enforce equilibrium, compatibility, yielding and the complementarity conditions. As a result of the disjunctive nature of the latter, the aforementioned optimization problem is reformulated as a NLP problem by using a penalty function method which performs satisfactorily in terms of robustness and efficiency. However, it is worth mentioning that the NLP problem is parameter and problem dependent and thus the proper choice of initial values of variables as well as setting the lower and upper bounds may be decisive in achieving the desirable convergence.

The optimization problem can incorporate various yield criteria in linearized form. In this paper, the generalized Gendy-Saleeb yield criterion is adopted that accounts for the axial-shear force-bending moment interaction. Numerical results highlight that shear force effect in yield condition may be significant as it leads to reduction of the load carrying capacity and thus to unsafe design, as compared to axial force-bending moment interaction. Moreover, for more complex configurations this may lead also to different collapse mechanisms.

Finally, wrapping up the vertices of the behaviour of the entire structure or either the beams or columns with the corresponding convex hulls and controlling their overall shape may be very helpful in optimal design of these structures.

Acknowledgements

European Union
European Social Fund
OPERATIONAL PROGRAMME
EDUCATION AND LIFELONG LEARNING
investing in knowledge society
MINISTRY OF EDUCATION, LIFELONG LEARNING AND RELIGIOUS AFFAIRS
MANAGING AUTHORITY
Co- financed by Greece and the European Union
NSRF 2007-2013
EUROPEAN SOCIAL FUND

This research has been co-financed by the European Union (European Social Fund–ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: Heracleitus II. Investing in knowledge society through the European Social Fund.

References

[1] G. Maier, "A quadratic programming approach for certain classes of nonlinear structural problems", Meccanica, 3, 121-130, 1968.

[2] G. Maier, "A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes", Meccanica, 5, 54-66, 1970.

[3] O.D. Donato, G. Maier, "Mathematical programming methods for the inelastic analysis of reinforced concrete frames allowing for limited rotation capacity", International Journal for Numerical Methods in Engineering, 4, 307–29,1972.

[4] O.D. Donato, G. Maier, "Historical deformation analysis of elastoplastic structures as a parametric linear complementarity problem", Meccanica, 11, 166-171, 1976.

[5] G. Maier, S. Giacomini, F. Paterlini, "Combined elastoplastic and limit analysis via restricted basis linear programming", Computer Methods in Applied Mechanics and Engineering, 19, 21–48, 1979.

[6] R.W. Cottle, "Monotone solutions of the parametric linear complementarity problem", Mathematical Programming, 3, 210-224, 1972.

[7] I. Kaneko, "Piecewise linear elastic-plastic analysis", International journal for numerical methods in engineering, 14, 757-767, 1979.

[8] G. Maier, D.E. Grierson, M.J. Best, "Mathematical programming methods for deformation analysis at plastic collapse", Computers and Structures, 7, 599-612, 1977.

[9] Z.Q. Luo, J.S. Pang, D. Ralph, "Mathematical programs with equilibrium constraint", Cambridge University Press, 1996.

[10] R. R. Wakefield, F. Tin-Loi, "Mathematical programming and uniqueness in nonholonomic plasticity", Computers and Structures, 34, 477-483, 1990.

[11] F. Tin-Loi, "On the optimal plastic synthesis of frames", Engineering Optimization Journal, 16, 91-108, 1990.

[12] F. Tin-Loi, JS. Pang, "Elastoplastic analysis of structures with nonlinear hardening: A nonlinear complementarity approach", Computer Methods in Applied Mechanics and Engineering, 107, 299–312, 1993.

[13] M.C Ferris, F.Tin-Loi, "Limit analysis of frictional block assemblies as a mathematical problem with complementarity constraints", International Journal of Mechanical Sciences, 43, 209-224, 2001.

[14] R. Ardito, G. Cocchetti, G. Maier, "On structural safety assessment by load factor maximization in piecewise linear plasticity", European Journal of Mechanics, 27, 859-881, 2008.

[15] G. Cocchetti, G. Maier, "Elastic–plastic and limit-state analyses of frames with softening plastic–hinge models by mathematical programming", International Journal of Solids and Structures, 40, 7219–44, 2003.

[16] S. Tangaramvong, F. Tin-Loi, "A complementarity approach for elastoplastic analysis of strain softening frames under combined bending and axial force", Engineering Structures, 29, 742–53, 2007.

[17] S. Tangaramvong, F. Tin-Loi, "Simultaneous ultimate load and deformation analysis of strain softening frames under combined stresses", Engineering Structures, 30, 664–74, 2008.

[18] M. A. A. Skordeli, C.D. Bisbos, "Limit and shakedown analysis of 3D steel frames via approximate ellipsoidal yield surfaces", Engineering Structures, 32, 1556-1567, 2010.

[19] A.S. Gendy, A.F. Saleeb, "Generalized yield surface representations in the elastoplastic three-dimensional analysis of frames", Computers and Structures, 49, 351-362, 1992.

[20] G. Cocchetti, G. Maier, X.P. Shen, "Piecewise linear models for interfaces and mixed mode cohesive cracks", Computer Modeling in Engineering Sciences, 3 279-98, 2002.

[21] M. Fukushima, G.H. Lin, "Smoothing methods for Mathematical Programs with equilibrium constraints", 12th International Conference on Informatics Research for Development of Knowledge Society Infrastructure, IEEE, 2004.

[22] M.B.Wong, “Plastic analysis and design of steel structures”, Butterworth-Heinemann is an imprint of Elsevier, 2009.